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A NEW METHOD OF SOLVING THE OPTIMAL CONTROL PROBLEM FOR A PARTIALLY OBSERVABLE STOCHASTIC VOLTERRA PROCESS*

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A new method is proposed for solving the linear-quadratic problem of optimal control for a partially observable stochastic Volterra process. The method relies on the representation and optimal estimation of optimal control in the form of integrals over the observable process. The integrands are non-stochastic and are defined by some system of integral equations, which may be solved numerically in advance. The optimal control is constructed directly from observations. An example demonstrating the implementation of the method by computer is given.

Integral Volterra equations first arose in creep theory and they are the foundation of this theory /1, 2/. They include a fairly large class of equations with a memory /3-5/, which play a central role in control theory and in various applications. The theory of optimal control of Volterra equations is a natural outgrowth of the theory of controllable differentiable equations. Filtering and optimal control theory for stochastic integral equations is rapidly developing. The classical solution of the problem by the "separation principle" /6, 7/ reduces to solving the optimal filtering problem and the optimal control problem under complete information for some subsidiary controllable system. The optimal control of the original problem is obtained /8/ as a linear functional of the (mean-square) optimal estimate of the optimal trajectory of motion, which in its turn is the solution of a system of stochastic integral equations. Thus, in order to construct the optimal control at each instant of time t we need to solve a system of stochastic integral equations in the interval $[0, t]$.

1. Let $\{\Omega, \sigma, P\}$ be the probability space, $\{f_t, t \in [0, T]\}$ a stream of σ--algebras, $f_t \subset \sigma$, $(x(t), y(t))$ a partially observable stochastic parocess defined by the equations

$$x(t) = x_0 + \int_0^t (a_0(t, s) u(s) + a_1(t, s) x(s))\, ds + \int_0^t b(t, s)\, dw_1(s) \tag{1.1}$$

$$dy(t) = A(t) x(t-h)\, dt + B(t)\, dw_2(t), \quad x(s) = 0, \quad s < 0 \tag{1.2}$$

$$h \geqslant 0, \quad x(t) \in R^n, \quad y(t) \in R^m, \quad u(t) \in R^l, \quad \mathrm{M}x_0 = 0, \quad \mathrm{M}x_0x_0' = D_0,$$

Here $y(t)$ is the observable process, $x(t)$ is the unobservable process, $u(t)$ is the control, $w_1(t)$ and $w_2(t)$ are f-measurable mutually indpendent k_1 and k_2 dimensional Wiener processes respectively, and x_0 is a normal random variable independent of $w_1(t)$ and $w_2(t)$. The coefficients in these equations are appropriately dimensioned, non-stochastic, and piecewise-continuous. The control objective is to minimize the functional

$$J(u) = \mathrm{M}\left[x'(T) F x(T) + \int_0^T u'(t) N(t) u(t)\, dt\right] \tag{1.3}$$

Here $N(t)$ is a piecewise-continuous matrix which is positive definite uniformly in t and F is a non-negative definite matrix (both non-stochastic); the prime denotes the transpose.

Let f_t^y be the minimum σ-algebra generated by the process $y(s), s \leqslant t$, $\mathrm{M}_t^y = \mathrm{M}\{\cdot/f_t^y\}$.

An admissible control is an arbitrary f_t^y-measurable process $u(t)$ for which the system of Eqs.(1.1), (1.2) is solvable and $J(u) < \infty$.

Lemma 1. The optimal control of problem (1.1)-(1.3) is representable in the form

$$u_0(\tau) = \int_0^\tau Q_0(\tau, s)\, dy_0(s) \tag{1.4}$$

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where $Q_0(\tau, s)$ is a non-stochastic matrix, $y_0(s)$ is the solution of system (1.1) and (1.2) for $u = u_0$.

Proof. Let $R_1(t, \tau)$ be the resolvent of the kernel $a_1(t, \tau)$. Then

$$R_1(t, \tau) = a_1(t, \tau) + \int_\tau^t R_1(t, s) a_1(s, \tau) ds \tag{1.5}$$

Let

$$\psi_1(T, \tau, a_0(\cdot, \tau)) = a_0(T, \tau) + \int_\tau^T R_1(T, s) a_0(s, \tau) ds \tag{1.6}$$

As in /9, 10/ we can show that the optimal control of problem (1.1)-(1.3) has the form

$$u_0(\tau) = -N^{-1}(\tau) \psi_1(T, \tau, a_0(\cdot, \tau)) FM_\tau^{y_0} x_0(T) \tag{1.7}$$

We rewrite Eq.(1.1) in the form

$$x_0(t) = \eta_0(t) + \int_0^t a_1(t, s) x_0(s) ds \tag{1.8}$$

$$\eta_0(t) = x_0 + \int_0^t a_0(t, s) u_0(s) ds + \int_0^t b(t, s) dw_1(s)$$

We know /11/ that the mean-square optimal estimate $m(\tau) = M_\tau^{y_0} x_0(T)$, $\tau \leqslant T$ of $x_0(T)$ defined by Eq.(1.8) is obtained from the observations $y_0(s)$, $s \leqslant \tau$ in the form

$$m_0(\tau) = \int_0^\tau G_0(\tau, s) dy_0(s) \tag{1.9}$$

From (1.7) and (1.9) we obtain the equality (1.4) for

$$Q_0(\tau, s) = -N^{-1}(\tau) \psi_1(T, \tau, a_0(\cdot, \tau)) FG_0(\tau, s) \tag{1.10}$$

From the proof of Lemma 1 it follows that the matrices $Q_0(\tau, s)$ and $G_0(\tau, s)$ defining the optimal control (1.4) and the optimal estimate (1.9) are related by (1.10) (together with the subsidiary equalities (1.5) and (1.6)).

We will derive another relationship between the matrices $Q_0(\tau, s)$ and $G_0(\tau, s)$. This will require the following notation. Let

$$q_0(t, \tau) = \begin{cases} \int_\tau^t a_0(t, s) Q_0(s, \tau) ds, & t \geqslant \tau \\ 0, & t < \tau \end{cases} \tag{1.11}$$

$$P_0(t, \tau) = a_1(t, \tau) + q_0(t, \tau + h) A(\tau + h) \tag{1.12}$$

and $R_0(t, \tau)$ is the resolvent of the kernel $P_0(t, \tau)$, i.e.,

$$R_0(t, \tau) = P_0(t, \tau) + \int_\tau^t R_0(t, s) P_0(s, \tau) ds \tag{1.13}$$

For an arbitrary matrix $f(\tau)$, $\tau \in [s, t]$, let

$$\psi_0(t, s, f(\cdot)) = f(t) + \int_s^t R_0(t, \tau) f(\tau) d\tau$$

Let

$$\begin{aligned} R(t, \tau) = {} & \psi_0(t, 0, E) D_0 \psi_0'(\tau, 0, E) + \\ & \int_0^{t\wedge\tau} \psi_0(t, s, b(\cdot, s)) \psi_0'(\tau, s, b(\cdot, s)) ds + \\ & \int_0^{t\wedge\tau} \psi_0(t, s, q_0(\cdot, s)) B_0(s) \psi_0'(\tau, s, q_0(\cdot, s)) ds \end{aligned} \tag{1.14}$$

$$S_h(\tau) = R(T, \tau - h) A'(\tau) + \psi_0(T, \tau, q_0(\cdot, \tau)) B_0(\tau), \quad B_0 = BB' \tag{1.15}$$

$$\begin{aligned} K_h(s, \tau) = {} & A(s) R(s - h, \tau - h) A'(\tau) + \\ & B_0(s) \psi_0'(\tau - h, s, q_0(\cdot, s)) A'(\tau) + A(s) \psi_0(s - h, \tau, q_0(\cdot, \tau)) B_0(\tau) \end{aligned} \tag{1.16}$$

Lemma 2. The matrices $Q_0(\tau, s)$ and $G_0(\tau, s)$ are connected by the equation

$$G_0(t,\tau)B_0(\tau) = S_h(\tau) - \int_0^t G_0(t,s)K_h(s,\tau)\,ds, \qquad \tau \in [0,t] \tag{1.17}$$

Proof. Let $m(t)$ be a f_t^y-measurable process of the form (1.9) with an arbitrary kernel $G(t,s)$. Using (1.9), (1.2) and the easily verified relationship $M(x_0(T) - m_0(t))\,m'(t) = 0$, we obtain

$$\int_0^t [M(x_0(T)x_0'(\tau-h))A'(\tau)\,d\tau + M(x_0(T)\,dw_2'(\tau))B'(\tau)]\,G(t,\tau) = \tag{1.18}$$
$$\int_0^t\int_0^t G_0(t,s)\,[A(s)M(x_0(s-h)x_0'(\tau-h))A'(\tau)\,d\tau\,ds +$$
$$B(s)M(dw_2(s)x_0'(\tau-h))A'(\tau)\,d\tau + A(s)M(x_0(s-h)\,dw_2'(\tau))B'(\tau)\,ds] \times$$
$$G'(t,\tau) + \int_0^t G_0(t,\tau)B_0(\tau)G'(t,\tau)\,d\tau$$

Substituting (1.4) into (1.1) and using (1.2), we have

$$x_0(t) = x_0 + \int_0^t P_0(t,s)x_0(s)\,ds + \int_0^t b(t,s)\,dw_1(s) + \int_0^t q_0(t,s)B(s)\,dw_2(s)$$

Since $R_0(t,\tau)$ is the resolvent of the kernel $P_0(t,\tau)$, we have

$$x_0(t) = \psi_0(t,0,E)x_0 + \int_0^t \psi_0(t,\tau,b(\cdot,\tau))\,dw_1(\tau) +$$
$$\int_0^t \psi_0(t,\tau,q_0(\cdot,\tau))B(\tau)\,dw_2(\tau)$$

Thus,

$$M(x_0(t)x_0'(\tau)) = R(t,\tau)$$
$$M(x_0(t)\,dw_2'(\tau)) = \psi_0(t,\tau,q_0(\cdot,\tau))B(\tau)\,d\tau$$

Substituting (1.19) into (1.18) and noting that the kernel $G(t,\tau)$ is arbitrary, we obtain (1.17).

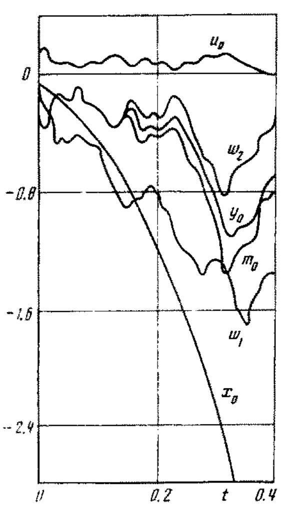

Fig.1

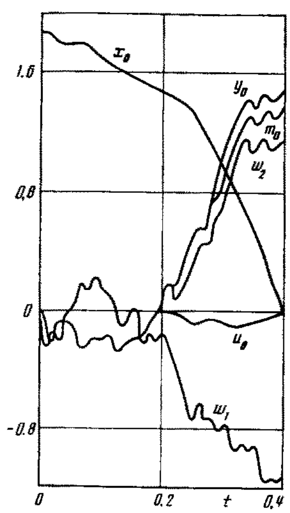

Fig.2

The relationships (1.10) and (1.17) may be regarded as a system of two equations in two unknowns. If the matrix $B_0(\tau)$ is invertible uniformly in τ, then (1.17) for each fixed t is a Fredholm equation of the second kind, which has a unique solution /11, 12/. Numerical methods of solving such equations are considered in /13/.

The system of Eqs. (1.10) and (1.17) is easily generalized to the case when Eq. (1.1) contains an unknown parameter and the noises in Eqs. (1.1) and (1.2) are dependent /14, 15/.

2. The algorithm for the numerical solution of the system of equations (1.10) and (1.17) is based on the method of successive approximations. Consider an arbitrary initial value of the matrix $G_0(t, s)$. Given $G_0(t, s)$, we apply the relationships (1.5), (1.6) and (1.10) to find the initial value of the matrix $Q_0(t, s)$. Given $Q_0(t, s)$ we apply the relationships (1.11)-(1.16) to construct Eq.(1.17), whose solution supplies the next approximation of the matrix $G_0(t, s)$. We then recalculate $Q_0(t, s)$ and so on, until two successive approximations agree (within specified accuracy limits). The algorithm was tested on some numerical examples.

Example. Consider the control problem

$$x^{\cdot\cdot}(t) = u + \sigma w_1^{\cdot}(t), \quad x(0) = x_0, \quad x^{\cdot}(0) = 0 \tag{2.1}$$

$$y^{\cdot}(t) = x(t-h) + w_2^{\cdot}(t), \quad x(s) = 0, \quad s < 0, \quad y(0) = 0 \tag{2.2}$$

$$J(u) = \mathrm{M}\left[x^2(T) + \int_0^T u^2(s)\,ds\right] \tag{2.3}$$

Eq.(2.1) after double integration takes the form (1.1). The system of Eqs.(1.10) and (1.17) corresponding to problem (2.1)-(2.3) was solved numerically by the above algorithms for $T = 0.4$, $\sigma = 10$, $\mathrm{M}x_0^2 = 1$. The normal random variable x_0 and the Wiener processes $w_1(t)$, and $w_2(t)$ were simulated by a normal pseudorandom number generator. The trajectories of the processes $x_0(t)$, $y_0(t)$, $u_0(t)$, $m_0(t)$ were constructed numerically using formulas (2.1), (2.2), (1.4) and (1.9). The numerical results for various samples of x_0, $w_1(t)$, $w_2(t)$ and various values of h are plotted in Fig.1 $(x_0 = -0.0865,\ h = 0)$ and Fig.2 $(x_0 = 1.856,\ h = 0.2)$.

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